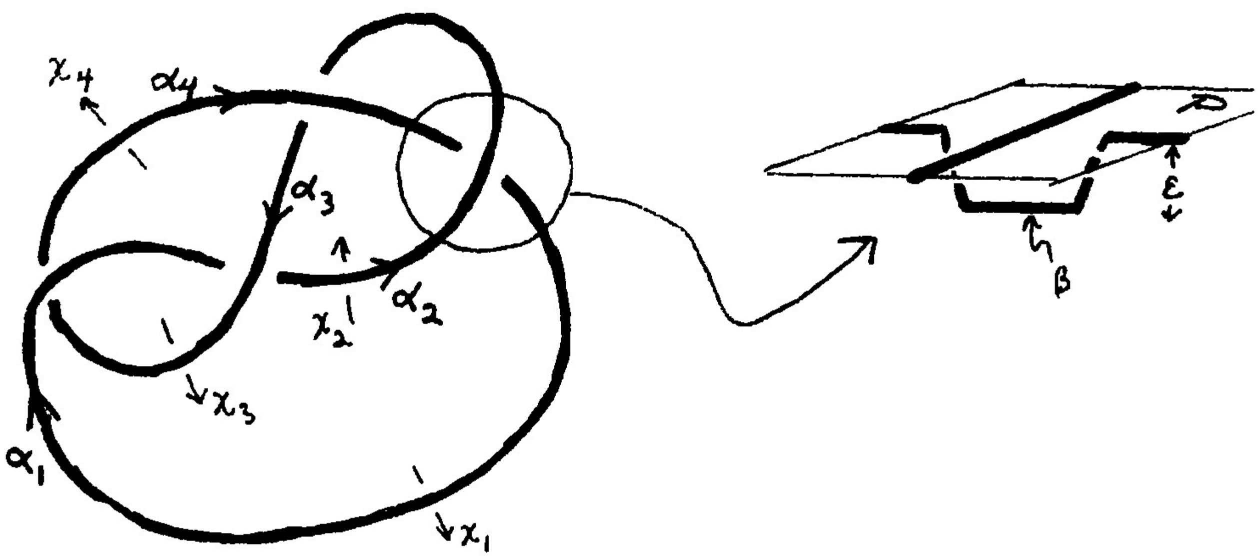
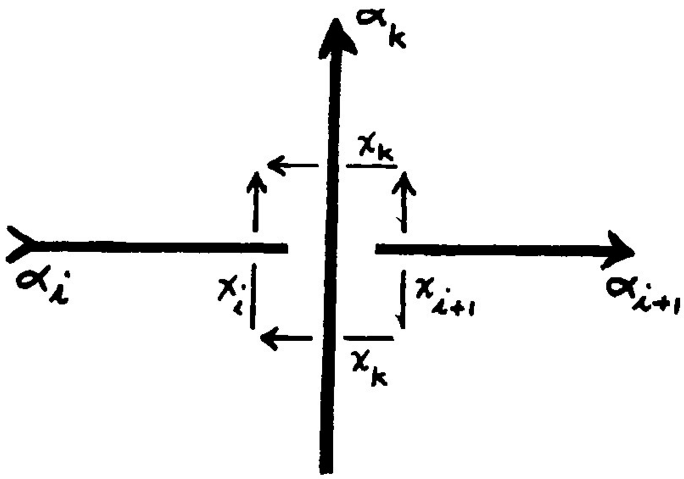
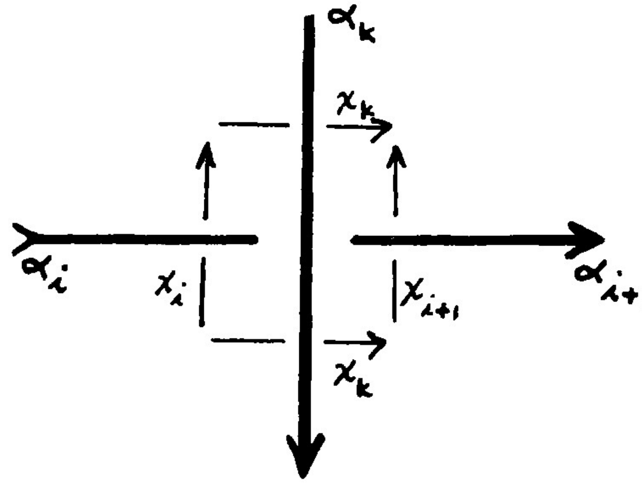
D. THE WIRTINGER PRESENTATION. This section describes a procedure for writing down a presentation of the group of a knot K in R^3, given a 'picture' of the knot. By a picture I mean a finite number of arcs $\alpha_1, \ldots, \alpha_n$ in a plane P (say, the x-y plane). Each α_i is assumed connected to α_{i-1} and α_{i+1} (mod n) by undercrossing arcs exactly as pictured below. The union of these is the knot K.



1. THE ALGORITHM. We assume for convenience that the α_i are oriented (assigned a direction) compatibly with the order of their subscripts. Draw a short arrow labelled x_i passing under each α_i in a right-left direction. This is supposed to represent a loop in $R^3 - K$ as follows. The point $(0, 0, 1) = *$ is taken as basepoint (best imagined as the eye of the viewer), and the loop consists of the oriented triangle from $*$ to the tail of x_i, along x_i to the head, thence back to $*$.

Now at each crossing, there is a certain relation among the x_i's which obviously must hold. The two possibilities are :





r_i : $\quad x_k x_i = x_{i+1} x_k \qquad\qquad x_i x_k = x_k x_{i+1}$

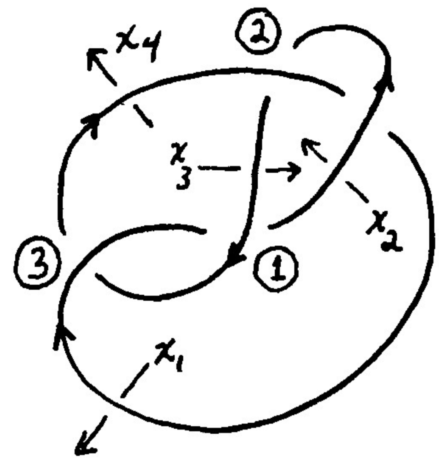
Here α_k is the arc passing over the gap from α_i to α_{i+1} ($k = i$ or $i+1$ is possible). Let r_i denote whichever of the two equations holds. In all, there are exactly n relations $r_1, \ldots, r_n$ which may be read off this way. We will see that these comprise a complete set of relations.

2. THEOREM : The group $\pi_1(R^3 - K)$ is generated by the (homotopy classes of the) x_i and has presentation

$$\pi_1(R^3 - K) = (x_1, \ldots, x_n; r_1, \ldots, r_n) .$$

Moreover, any one of the r_i may be omitted and the above remains true.

3. EXAMPLE : For the figure-eight knot, we have a presentation with generators x_1, x_2, x_3, x_4 and relations



(1) $x_1x_3 = x_3x_2$

(2) $x_4x_2 = x_3x_4$

(3) $x_3x_1 = x_1x_4$

We may simplify, using (1) and (3) to eliminate $x_2 = x_3^{-1}x_1x_3$ and $x_4 = x_1^{-1}x_3x_1$ and substitute into (2) to obtain the equivalent presentation

$$\pi_1(R^3 - \text{figure-eight}) \cong (x_1, x_3;\ x_1^{-1}x_3x_1x_3^{-1}x_1x_3 = x_3x_1^{-1}x_3x_1) .$$

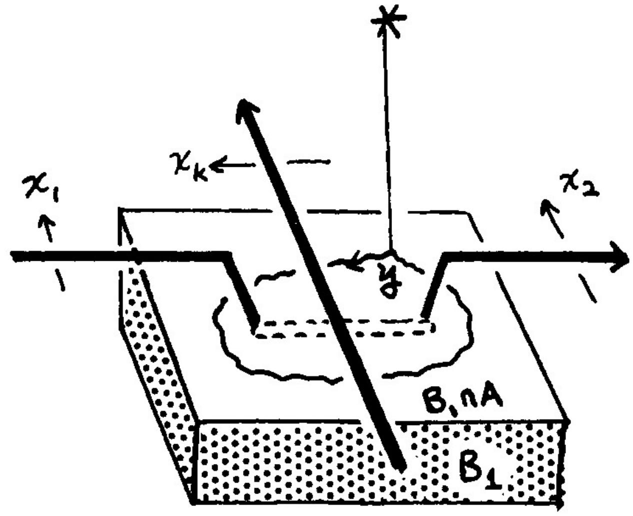
4. EXERCISE : Show combinatorially that the fourth relation $(x_2x_4 = x_1x_2)$ is a consequence of the other three.

5. EXERCISE : Verify that the figure-eight knot is nontrivial. (Try mapping its group onto a nonabelian finite group, as in lemma B4).

6. PROOF OF THEOREM 2 : Recall that K lies in the plane $P = \{z = 0\}$ of R^3, except where it dips down by a distance ε at each crossing. In order to apply Van Kampen's theorem, we dissect $X = R^3 - K$ into $n+2$ pieces $A, B_1, \ldots, B_n$, and C. Let

$$A = \{z \geq -\varepsilon\} - K .$$

The lower boundary of A is the plane $P' = \{z = -\varepsilon\}$ with n line segments $\beta_1, \ldots, \beta_n$ removed. Let B_i be a solid rectangular box whose top fits on P' and surrounds β_i. But we remove β_i itself from B_i, and (in order that B_i contain $*$) adjoin an arc running from the top, straight to $*$, missing K. The B_i may be taken to be disjoint from one another. Finally, let



$$C = \text{the closure of everything below } A \cup B_1 \cup \ldots \cup B_n\,,$$
$$\text{plus an arc to } * .$$

7. EXERCISE : Verify that $\pi_1(A)$ is a free group generated by $x_1, \ldots, x_n$.

Now we investigate the effect of adjoining B_1 to A. B_1 itself is simply-connected, and $B_1 \cap A$ is a rectangle minus β_1, plus the arc to $*$, so $\pi_1(B_1 \cap A)$ is infinite cyclic, with generator y. As is clear from the picture, when y is included in A, it becomes the word $x_k x_1^{-1} x_k^{-1} x_2$ (here we are assuming the crossing is of the first type). Thus, by Van Kampen, $\pi_1(A \cup B_1)$ has generators $x_1, \ldots, x_n$ and the single relation $x_k x_1^{-1} x_k^{-1} x_2 = 1$. This is equivalent to $x_k x_1 = x_2 x_k$, which is r_1 . Thus

$$\pi_1(A \cup B_1) = (x_1, \ldots, x_n;\ r_1)\ .$$

Similarly, adjoining B_2 , we argue that

$$\pi_1(A \cup B_1 \cup B_2) = (x_1, \ldots, x_n; r_1, r_2) ,$$

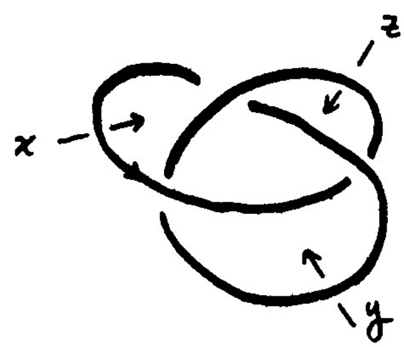
et cetera, so that

$$\pi_1(A \cup B_1 \cup \ldots \cup B_n) = (x_1, \ldots, x_n; r_1, \ldots, r_n) .$$

Finally, adjoining C to $A \cup B_1 \cup \ldots \cup B_n$ has no effect on the fundamental group, since both C and $C \cap (A \cup B_1 \cup \ldots \cup B_n)$ are simply-connected.

This completes the proof, except for the observation that one of the r_i (say r_n) is redundant. To see this, work in $S^3 = R^3 + \infty$. Let $A' = A + \infty$ and $C' = B_n \cup C + \infty$. It is clear that $A' \cup B_1 \cup \ldots \cup B_{n-1} \cup C' = S^3 - K$, $\pi_1(A') = \pi_1(A)$, and adjoining $B_1, \ldots, B_{n-1}$ has the same effect as before. But now we note that $C' \cap (A' \cup B_1 \cup \ldots \cup B_{n-1})$ is simply-connected, being a 2-sphere minus an arc, and so is C' . Thus we reach the same conclusion without adjoining the relation r_n .

8. EXAMPLE : We recompute the group of the trefoil using the Wirtinger method.

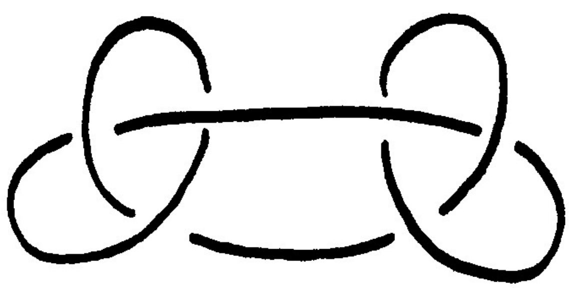


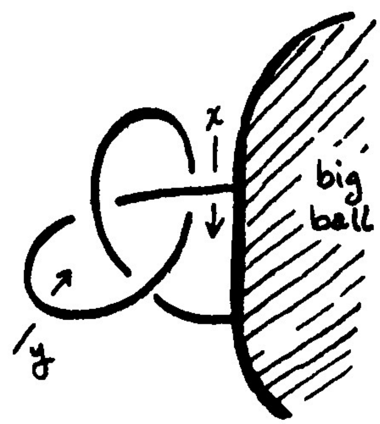
We have generators x, y, z and relations $xz = zy$, $yx = xz$. The second may be used to eliminate $z = x^{-1}yx$, which converts the first relation to $yx = x^{-1}yxy$. Thus we have another presentation for the trefoil group

$$G_{2,3} = (x, y; xyx = yxy) .$$

9. EXERCISE : Show directly that this is equivalent to the presentation $(a, b; a^2 = b^3)$.

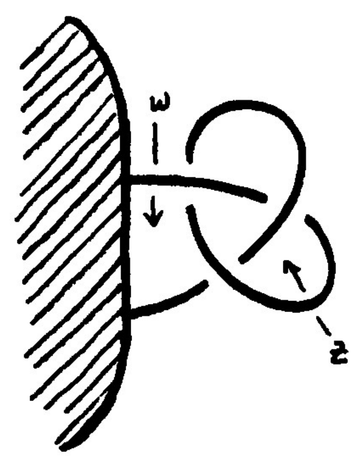
10. EXAMPLE : The square knot





We may use a short-cut by considering the complement of :

It is clear that this has the homotopy type of the complement of the trefoil. Likewise for the complement of :



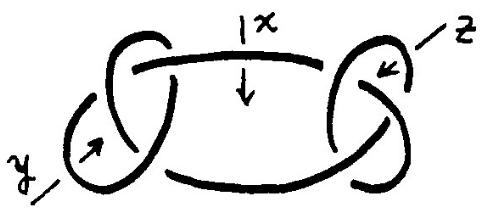
The union of these complements gives the complement of the square knot, so we use Van Kampen's theorem to see that :

Group of the square knot

$= (x, y, w, z; xyx=yxy, wzw = zwz, x = w)$

$= (x, y, z; xyx = yxy, xzx = zxz).$

11. EXAMPLE :



Group of the <u>Granny</u> <u>knot</u>

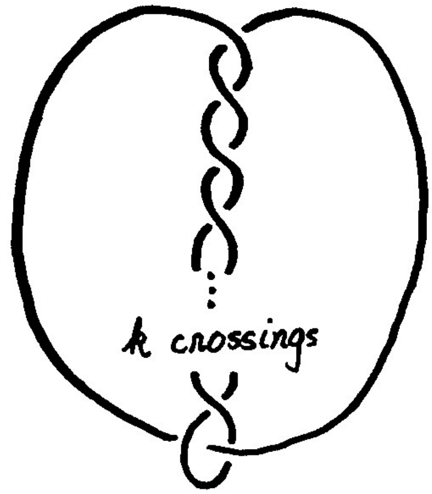
$= (x, y, z; xyx = yxy, xzx = zxz)$

is isomorphic with the group of the square knot.

It happens that, in fact, the square and granny knots are <u>not</u> equivalent although the methods we have discussed so far do not distinguish them. When we do this (see 8E15) we will have established

12. THEOREM : The group of a knot is not a complete knot invariant (that is, $K \rightsquigarrow \pi_1(R^3 - K)$ is not one-to-one).

13. NOTE : The complements of the square and granny knots are actually not homeomorphic, as is shown by Fox [1952] using "peripheral structure" of π_1



14. EXERCISE : Compute a presentation for the group of the knot shown here. Show that it has a presentation with only two generators and one relation.

15. REMARK : Given two knot group presentations, it is often quite difficult to prove that they present non-isomorphic groups. Later we will develop knot invariants which are much more readily compared to distinguish knots and links.

E. REGULAR PROJECTIONS. The usual way of describing a knot is by drawing a picture, as described above. That this algorithm applies to arbitrary tame knots, is the object of the following exercises.

Let K be a polygonal knot in R^3. Let P be any plane and $p : R^3 \longrightarrow P$ the orthogonal projection. Say that P is <u>regular</u> for K provided every $p^{-1}(x)$, $x \in P$, intersects K in 0, 1 or 2 points and, if 2, neither of them is a vertex of K.

1. EXERCISE : Given any polygonal K and plane P, one can make P regular for K by arbitrarily small perturbations of either P or K.

2 EXERCISE : Let K be a polygonal knot with vertices $v_o, \ldots, v_r$. Then there exists a positive number $\varepsilon = \varepsilon(K)$ such that whenever $v'_o, \ldots, v'_r$ are points in R^3 with $|v_i - v'_i| < \varepsilon$ for all i, the polygon $K' = v'_o v'_1 \ldots v'_r v'_o$ is also a knot, and is ambient isotopic to K .

3. EXERCISE : If P is regular for K , then K is ambient isotopic to a knot of the type described in the section on the Wirtinger presentation.

4. DEFINITION : The <u>deficiency</u> of a group presentation equals the number of generators minus the number of relations.

5. COROLLARY : Every tame knot group has a finite presentation of deficiency one

6. EXERCISE : Use the Wirtinger algorithm to prove that the abelianization of any tame knot group is infinite cyclic. (This also follows from exercise 2E6 , since π_1 abelianized is H_1 and holds for wild knots as well.)

7. EXERCISE : Show that no (tame) knot group has a presentation with deficiency two.

8. EXERCISE : Every tame knot in R^3 possesses a tubular neighbourhood, which itself is equivalent to a polyhedron in R^3. Components of a tame link in R^3 have disjoint tubular neighbourhoods.